A comparative study of Numerical Solutions of heat and advection-diffusion equation

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Abstract: A comparative study of Numerical Solutions of One Dimensional heat and advection-diffusion equation is obtained by collocation method. Equations are solved numerically by using Orthogonal collocation on finite elements. Numerical values obtained are in good agreement with exact ones. It can be seen that the method of orthogonal collocation on finite elements give better results. Collocation points are taken to be the roots of shifted Legendre polynomial. Lagrange basis are used to describe the equation. The results are verified for three test problems. The system of differential algebraic solutions is obtained and is solved using MATLAB ode15s. The results are examined in terms of absolute error.

Keywords: Collocation, Lagrange basis, advection-diffusion Shifted Legendre polynomials.

Introduction:

Partial differential equations are used for modeling of various diverse physical applications and problems such as heat flow, sound waves propagation, ball vibration, ink diffusion in water, electro-magnetic fields, hydrogen atom's quantum behavior. Therefore, various applications of Partial differential equations in the field of applied mathematics play very important role and lead to solution of real life problems. Most applications of partial differential equations in science and engineering require numerical solutions, since the equations are typically too complicated, both in number and form, to admit analytical solutions.

In engineering, physical and earth sciences, substance transportation is known as advection. For example, advection is the downstream transport of pollutants or silt in a river by bulk flow of water. When molecules moves randomly then matter is also transported from one part of a system to another, this process of transportation is diffusion. The advection-diffusion equation is a combination of the diffusion and advection equation and describes the phenomenon where particles, energy or other physical quantities are transferred inside a physical system due to two processes diffusion and advection. Environmental pollution problems can always be reduced through numerical solution of advection-diffusion based mathematical model.

In case of water pollution, the degradation of hydro-environment is due to the deposition of immiscible solute and heavy metals in water bodies through advection and diffusion processes. Due to industrialization and increase in population, the pollution is increased very much in recent few decades which draw the attention of scientists of various fields. Therefore Advection-diffusion equation is used by environmentalists, engineers, hydrologists and mathematical modellers to describe the pollutant concentration with respect to time and position. The numerical and analytical solution of such partial differential equations are useful to check the level of pollutants which starts affecting the habitat health. Therefore, these solutions help in maintaining the environment and timely action against the pollution.

It has been used to describe atmospheric pollution, solute contamination in the liquid flowing through tubes [1], dispersion in porous media [2], in various ranges of engineering, having various applications in industry [3], in water transfer in soil [4], in dispersion of dissolved salts underground water [5]. During recent years, considerable efforts have been made for the numerical and analytic solutions of different kinds of advection – diffusion equations by the different methods.

Sun and Zhang (2003), proposed a class of new finite difference schemes, compact boundary value method (CBVM) to solve the one dimensional heat equation with high-order accuracy and stability [6]. Chen et. al. (2003) discussed the transportation of solute in a radially convergent flow field with scale dependent dispersion through a novel mathematical model [7]. Meerschaert and Tadjeran (2004) developed the mathematical model for the transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology using the one dimensional fractional advection-dispersion equations with variable coefficients on a finite domain [8].

Allhumaizi (2006) developed and applied the moving collocation method to simulate the dynamics of a short time convection-diffusion-reaction model [9]. The solutions of the one and two dimensional, steady state and time dependent advection-diffusion equation through ADMM (Advection Diffusion Multilayer Model) Model has been reported by Moreira et. al. (2006) [10]. La Rocca and Power (2008) proposed the double boundary collocation approach based on the meshless radial basis function Hermitian method and compared with the conventional single collocation [11]. A high order method for solving the one dimensional heat and advection equation has been proposed by Mohebbi & Dhghan(2010)[12].

Jaiswal and Kumar (2011), used one dimensional advection-dispersion equation with variable coefficients to obtain analytical solutions in two cases, first one the solute dispersion is time dependent and second is dispersion and the velocity both have spatially dependent expressions [13]. Ahmed (2012) solved the advection-diffusion equation with constant and variable coefficients using a new finite difference equation as well as a numerical scheme [14]. Goh *et. al*. (2012) solved the one-dimensional heat and advection-diffusion equations with accuracy and stability using the numerical method based on the cubic B-spline collocation [15].

Gurarslan *et al.* (2013) numerically solve the one-dimensional advection-diffusion equation using a sixth order compact difference scheme and fourth order Runge-Kutta scheme in space and time respectively [16]. Arora and Kaur (2015), proposed the orthogonal collocation technique on finite elements to numerically solve the heat conduction problems [17].

Mathematical formulation:

One dimensional linear advection-diffusion equation in physics has been modeled mathematically and can be written as:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left[D(x,t)\frac{\partial c}{\partial x} - u(x,t)c\right] \qquad (1)$$

where c is solute concentration, x is space variable,t is time, D(x,t) is solute diffusion and is known as diffusion co efficient , if it is uniform and steady and u(x,t) is velocity of the medium.

If the medium is porous then it is derived from the principle of conservation of mass and Fick's law of diffusion and also velocity of flow satisfies the Darcy's law.

Therefore consider,

$D(x,t)=D_0\, g_1(x,t)$ and $u(x,t)= u_0\, g_2(x,t)$

By (1) we have:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left[D_0\, g_1(x,t)\frac{\partial c}{\partial x} - u_0\, g_2(x,t)c\right]$$

Next consider a new independent variable, X. Thus for X we have

$$\frac{\partial X}{\partial x} = \frac{-1}{g_1(x,t)} \qquad X = -\int \frac{dx}{g_1(x,t)}$$

$$\frac{\partial c}{\partial t} = D_0\left[\frac{\partial c}{\partial x}\left[\frac{\partial}{\partial x} g_1(x,t)\right] + g_1(x,t)\frac{\partial^2 c}{\partial x^2}\right] - u_0\frac{\partial}{\partial x} g_2(x,t)C$$

$$\frac{\partial X}{\partial x} = \frac{-1}{g_1(x,t)}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial c}{\partial x}\left[\frac{-1}{g_1(x,t)}\right]$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial c}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{-1}{g_1(x,t)}\,\frac{\partial c}{\partial X}\right]$$

$$= \frac{-1}{g_1(x,t)}\left[\frac{\partial^2 c}{\partial x^2}\cdot\frac{\partial X}{\partial x}\right] + \left[\frac{1}{g_1^2(x,t)}\,\frac{\partial c}{\partial X}\right]$$

$$\frac{\partial^2 c}{\partial x^2} = \left[\frac{-1}{g_1^2(x,t)}\,\frac{\partial^2 c}{\partial X^2} + \left[\frac{1}{g_1^2(x,t)}\,\frac{\partial c}{\partial X}\right]\right]$$

$$\frac{\partial c}{\partial t} = D_0\left[\frac{-g_1^1(x,t)}{g_1(x,t)}\,\frac{\partial c}{\partial X} + \frac{g_1(x,t)}{g_1^2(x,t)}\left\{\frac{\partial^2 c}{\partial X^2} + \frac{\partial c}{\partial X}\right\}\right] - u_0\left[\frac{\partial}{\partial X}\,[g_2(x,t)c]\right]\cdot\frac{-1}{g_1(x,t)}$$

$$g_1(x,t)\frac{\partial c}{\partial t} = D_0\left[\frac{\partial^2 c}{\partial X^2} + \frac{\partial c}{\partial X} - \frac{\partial c}{\partial X}\right] + \frac{\partial}{\partial x}\,[g_2(x,t)c]$$

$$g_1(x,t)\frac{\partial c}{\partial t} = D_0\frac{\partial^2 c}{\partial X^2} + u_0\frac{\partial}{\partial x}\,[g_2(x,t)c] \qquad (2)$$

Next consider a time dependent diffusion along a uniform flow i.e

Let $\quad g_1(x,t) = g(t)$

and $\quad g_2(x,t) = 1$

where g(t) is chosen such that g(t)=1 for t=0. This g(t) is in non-dimensional form of variable t.

Thus $X = -\int \frac{dx}{g_1(x,t)}$ implies $X = \frac{-x}{g(t)}$

Therefore by (2) we have;

$$g(t)\frac{\partial c}{\partial t} = D_0\frac{\partial^2 c}{\partial X^2} + u_0\frac{\partial c}{\partial x} \qquad (3)$$

By using $\frac{\partial T}{\partial t} = \frac{1}{g(t)}$, this gives $\frac{\partial c}{\partial t} = \frac{\partial c}{\partial T}\cdot\frac{\partial T}{\partial t} = \frac{1}{g(t)}\,\frac{\partial c}{\partial T}$

Thus by (3) we have

$\frac{\partial c}{\partial T} = D_0\frac{\partial^2 c}{\partial X^2} + u_0\cdot\frac{\partial C}{\partial X}$, which is a advection –diffusion equation with constant coefficient.

NUMERICAL TECHNIQUE:

Weighted residual method is one of the numerical techniques which is widely used to find an approximate solution of the boundary value problems. In this method, an approximate solution is substituted into the differential equation to form the residual and then residual is set orthogonal to the weight function of the approximating polynomial. Galerkin method, Moment method, Ritz method, Least Square method, Collocation methods etc. all are types of weighted residual method.

In case of stiff boundary value problems orthogonal collocation method does not give appropriate results for large values of different parameters. Paterson and Cresswell improve the method of orthogonal collocation by implement in it the properties of method of finite elements, which then known as method of orthogonal collocation on finite elements. This numerical technique was further

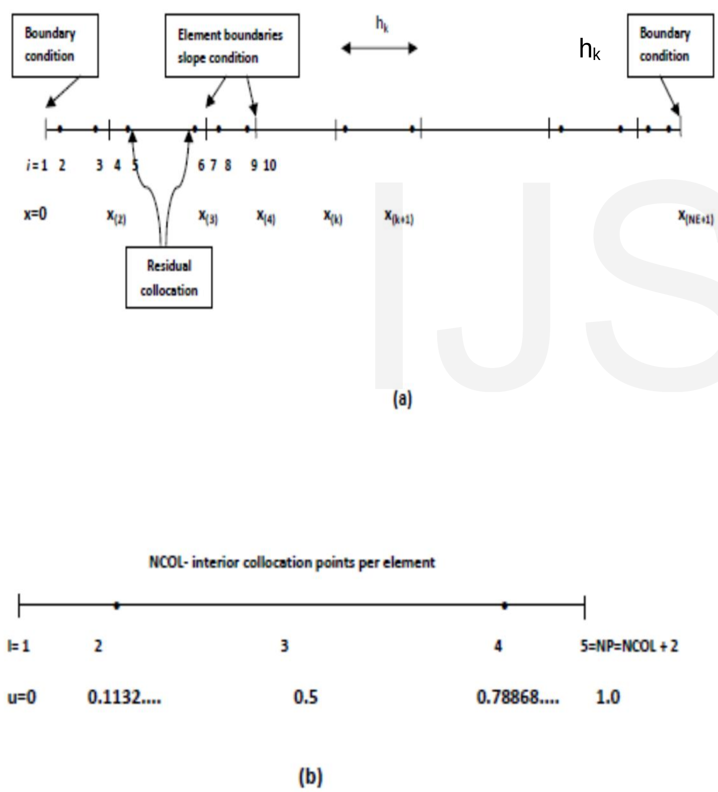


Figure: Collocation points on finite elements, lagrangian cubic polynomials. (a) Global numbering system *i.* (b) local numbering system.

extended by Carey and Finlayson specifically to solve the problems deals with large Thiele modulus. In OCFE the space domain is divided into sub –domains called finite elements and then method of orthogonal collocation is applied within each sub-domain fulfilling the essential condition of continuity of function and its derivative of first order at the boundaries of the elements. In order to apply collocation within each element, a new variable u is introduced in each element $[x_\ell, x_{\ell+1}]$ in such a way that as x varies from x_ℓ to $x_{\ell+1}$, v varies from 0 to 1, i.e., $v = \dfrac{x - x_\ell}{x_{\ell+1} - x_\ell}$. By applying the orthogonal collocation directly on v within each element, one gets the collocation equations in terms of the solutions at the collocation points. The representation of location of collocation points for OCFE is as under

INTERPOLATION POLYNOMIAL

The polynomial interpolation means to construct such a polynomial that interpolates the given n+1data points. There are several techniques to construct interpolating polynomials. One of the major technique is Lagranges interpolation. Lagrangian interpolating polynomials is given by $L(x) = \sum_{i=0}^{n} \frac{g(x_i) l(x)}{(x - x_i) l^1(x_i)}$

where $l(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$

In orthogonal collocation method the trial function is approximated in terms of Lagrangian interpolation polynomial, where xi's are the zeros of the orthogonal polynomial $P_n(x)$, $x_1=0$ and $x_{n+1}=1$. The discretization matrices for first and second order derivative of approximating function at j^{th} collocation point are obtained by differentiating the interpolating polynomial at j^{th} collocation point.

COLLOCATION POINT

The convergence of any numerical technique is highly dependent on the selection of collocation points. In this study the roots of Legendre Polynomial are taken as collocation points, which is a special case of Jacobi polynomial . In case of Legendre polynomial, 0 and 1 are taken to be the boundary points. The legendre polynomials are the solution of legendre equation.

A recurrence relation giving Legendre polynomial is given by:

$$P_{n+1}(cos\theta) = cos\theta\, P_n(cos\theta) - \frac{1 - cos^2\theta\, d(P_n(cos\theta))}{(n+1)d(cos\theta)}$$

Taking $t = cos\theta$ we get:

$$P_{n+1}(t) = t\, P_n(t) - \frac{1 - t^2 P_n^i(t)}{(n+1)}$$

Consider $P_0(t) = 1$ as a starting condition, then applying above repeatedly we get:

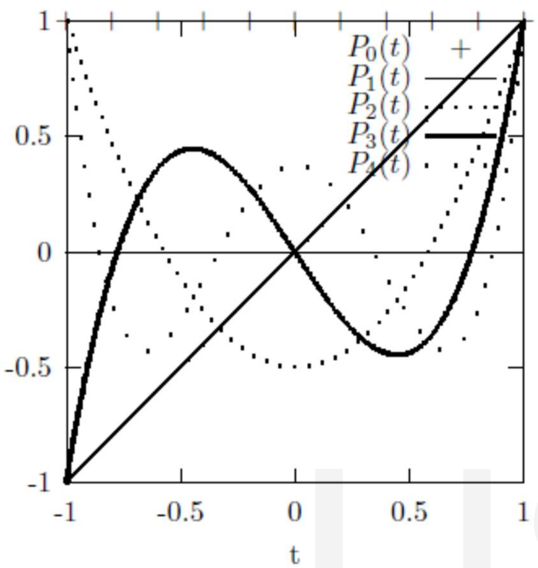
$$P_1(t) = t$$

$$P_2(t) = \frac{3t^2 - 1}{2}$$

$$P_3(t) = \frac{5t^3 - 3t}{2}$$

$$P_4(t) = \frac{35t^4 - 30t^2 + 3}{8}$$

Which are collectively represented graphically as follows:



The xj's are transformed onto the interval [0,1] using the formula given by,

$$u_{n+3-j} = \frac{x_j}{2} + \frac{1}{2}$$

where u_j is the local variable and x_j is the global variable.

ERROR ANALYSIS

Let g(x) is any (n+1) times continuously differentiable function then for $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$, and for any kind of (n+1) knots L(x) be the nth order Lagrange polynomial which are choosen such as $g(x_i) = L(x_i)$,i=0,1,2,...,n then following results holds.

LEMMA 1: Let g and L are choosen such as they satisfies all their respective properties described above then for all integers i , $0 \le i \le n$,the following result holds

$$\|g^i - L^i\| \le \|l^i\| \frac{\|g^{n+1}\|}{(n+1)!}$$

Where $l(x) = (x - x_0)(x - x_1) \dots (x - x_n)$ and where $\|.\|$ denotes the supremum norm on [a,b].

LEMMA 2: If lemma 1 is true then it is always true for the case of repeated roots.

Lemma 3: $|g(x) - L(x)| \le l(x) \frac{\|g^{n+1}\|}{(n+1)!}$

Lemma 4: For L(x) = L[g,x], lagrange interpolation polynomial of degree n satisfies $g(x_i) = L(x_i)$,

$$i = 0,1,2, \dots n, then$$

$$\int_0^1 \frac{|K^{(i,0)}(x,t)|}{n!} dt = sup \frac{(|g^i(x) - L^i[g,x]|)}{\|g^{(n+1)}\|},$$

$$g \in C^{(n+1)}[0,1]$$

where $K(x,t) = (x-t)_+^n - L[(x-t)_+^n, x]$

and

$$(x-t)_+^n = \begin{cases} (x-t)^+ & for\ x - t \ge 0 \\ 0 & for\ x - t < 0 \end{cases}$$

Some Identities which are used by Curry and Schoenberg:

1) $\int_0^1 \frac{K(x,t)}{n!} = \frac{l(x)}{(n+1)!}$
2) $M(t, x_0, x_1, \dots, x_n) = \sum_{i=0}^{n} \frac{n(x_i - t)_+^{n-1}}{l(x_i)}$
3) $M(t) > 0$, $if\ 0 < t < 1$
4) M(t) = 0 if $t \le 0\ or\ t \ge 1$
5) $\int_0^1 M(t) dt = 1$

Theorem: For $g(x) \in C^{(n+1)}[a,b]$ and $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$

and $g(x_i) = L(x_i)$,i=0,1,2,...,n, then

$$\|g^i - L^i\| \le \|l^i\| \frac{\|g^{n+1}\|}{(n+1)!}$$

Also this gives best possible result when g(x is substituted by l(x)

Proof: From identities 2 and 3 if a<b, then we have

$$0 < \int_a^b M(t, x_0, x_1, \dots, x_n)\, dt$$
$$= -\sum_{i=0}^{n} \frac{(x_i - b)_+^n}{l^i(x_i)} + \sum_{i=0}^{n} \frac{(x_i - a)_+^n}{l^i(x_i)}$$

Thus consider V(t)=$\sum_{i=0}^{n} \frac{(x_i - t)_+^n}{l^i(x_i)}$ is a monotonic decreasing function and is strictly decreasing for 0<t<1. Now since by lemma 5, we have

$$|g^n(x) - L^n[g,x]|$$
$$= \left| \int_0^1 \left((x-t)_+^0 - \sum_{i=0}^{n} \frac{(x_i - t)_+^n}{l^i(x_i)} \right) dt \right| * \|g^{n+1}(t)\|$$
$$\le \int_0^1 \left| (x-t)_+^0 - \sum_{i=0}^{n} \frac{(x_i - t)_+^n}{l^i(x_i)} \right| dt * \|g^{n+1}(t)\|$$

$= \int_0^1 |(x-t)_+^0 - V(t)|\, dt * \|g^{n+1}(t)\|$

Define

$$C_n(x) = \int_0^1 |(x-t)_+^0 - V(t)|\, dt$$

$$(x-t)_+^0 = \begin{cases} (x-t)^0 & for\ x-t \geq 0 \\ 0 & for\ x-t < 0 \end{cases}$$

Implies

$$(x-t)_+^0 = \begin{cases} 1 & for\ 0 \leq t \leq x \\ 0 & for\ x \leq t \leq 1 \end{cases}$$

$$C_n(x) = \int_0^1 |(x-t)_+^0 - V(t)|\, dt$$

$$= \int_0^x [1 - V(t)]\, dt + \int_x^1 [\, V(t)]\, dt$$

Gives

$$C_n^I(x) = V(0) + V(1) - 2V(x)$$

Since, $V(0) = V(1) = 1$

Implies
$C_n^I(x) = 2 - 2V(x) = 2\big(1 - V(x)\big)$

and therefore $C_n^{II}(x) = -2\,V^I$(x) > 0 and V(x) is decreasing function in 0<t<1.

$\|C_n\| = \max\{C_n(0), C_n(1)\}$

$$= \max \int_0^1 \frac{|K^{(n,0)}(x,t)|}{n!}\, dt \quad \text{for} 0 \leq x \leq 1$$

Since by identity (1)

$$\int_0^1 \frac{K^{(n,0)}(x,t)}{n!} = \frac{l^n(x)}{(n+1)!}$$

$$\|C_n\| = \max\left\{-\int_0^1 \frac{K^{(n,0)}(0,t)}{n!}, \int_0^1 \frac{K^{(n,0)}(1,t)}{n!}\right\}$$

Since $K^{(n,0)}(0,t) < 0\ and\ K^{(n,0)}(1,t) > 0$

$$\|C_n\| = \max\left\{-\frac{l^n(0)}{(n+1)!}, \frac{l^n(1)}{(n+1)!}\right\}$$

and from identity (5)

$|g^n(x) - L^n[g,x]| \leq \int_0^1 \left|(x-t)_+^0 - \sum_{i=0}^n \frac{(x_i-t)_+^n}{l^i(x_i)}\right| dt *$
$\|g^{n+1}(t)\|$

$$= \int_0^1 \left|K^{(i,0)}(x,t)\right| dt \left\|\frac{g^{n+1}(t)}{n!}\right\|$$

$$= \|C_n\| \left\|g^{(n+1)}\right\|$$

Thus,

$$|g^n(x) - L^n[g,x]| = \|C_n\| * \|g^{n+1}(t)\|$$

$$\leq \max\left\{-\frac{l^n(0)}{(n+1)!}, \frac{l^n(1)}{(n+1)!}\right\} \|g^{n+1}(t)\|$$

$$\leq \left\|\frac{l^n}{(n+1)!}\right\| \|g^{n+1}\|$$

and for g(x) =l(x) it gives best possible bound.

Problem1: Consider one dimensional heat advection equation

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2} \qquad for\ 0 < x < 1, t > 0$$

with initial condition

$$C(x,0) = \sin(\pi x)$$

and boundary conditions

C (0,t) = C(1,t) = 0.

The exact solution of above equation is given by $C(x,t) = \exp(-\pi^2 t)\sin(\pi x)$.The above problem is solved by using method of orthogonal collocation on finite elements using lagranges basis and compared for different values of h which are shown in table1. The collocation points are taken as roots of Legendre Polynomial described as above. The error analysis in terms of absolute error for above problem for different h is given in table 2.

Table1: Comparison of numerical values for different values of h for problem1

h=1/12	h=1/20	h=1/40	h=1/80	h=1/40 (7 int points)
1.3396e-01	2.4108e-01	3.7686e-02	3.9226e-02	1.9028e-01
4.9938e-02	8.9860e-02	1.4030e-02	1.2922e-01	7.0917e-02
1.8631e-02	3.3492e-02	5.2314e-03	4.8161e-02	2.6432e-02
6.9523e-03	1.2476e-02	1.9471e-03	1.7950e-02	9.8532e-03
2.5903e-03	4.6488e-03	7.2475e-04	6.6902e-03	3.6727e-03
9.6468e-04	1.7330e-03	2.6967e-04	2.4935e-03	1.3687e-03
3.5927e-04	6.4616e-04	1.0037e-04	9.2934e-04	5.1007e-04
1.3360e-04	2.4097e-04	3.7331e-05	3.4637e-04	1.9006e-04
4.9657e-05	8.9975e-05	1.3805e-05	1.2910e-04	7.0798e-05
1.8227e-05	3.3624e-05	5.1068e-06	4.8115e-05	2.6360e-05
6.6992e-06	1.2681e-05	1.8358e-06	1.7933e-05	9.7862e-06
2.4036e-06	4.7673e-06	6.5468e-07	6.6837e-06	3.6369e-06
7.7443e-07	1.7751e-06	2.3056e-07	2.4911e-06	1.3588e-06
2.5198e-07	5.9936e-07	1.1673e-07	9.2844e-07	5.2429e-07
7.5646e-08	2.0581e-07	6.3816e-08	3.4604e-07	1.9753e-07
4.1827e-08	8.0190e-08	2.2280e-08	1.2897e-07	7.3387e-08
6.9681e-08	5.5539e-08	9.6713e-09	4.8068e-08	5.5355e-08
6.3687e-08	1.2540e-07	7.5104e-08	1.7915e-08	5.4059e-08
2.3885e-08	2.2173e-07	9.0196e-08	6.6772e-09	5.3789e-08
2.4894e-09	1.5009e-07	1.4775e-07	2.4887e-09	3.5939e-08
2.2837e-09	5.9066e-08	9.7093e-08	9.2754e-10	8.0696e-09
5.0996e-09	2.0171e-08	6.3450e-08	3.4570e-10	7.2787e-09
7.0039e-09	4.2589e-08	4.8376e-09	1.2885e-10	1.0558e-08
1.5587e-09	3.0422e-08	2.8951e-08	4.8023e-11	3.4881e-09
9.6295e-10	2.2610e-09	3.0929e-08	1.7899e-11	2.5487e-09
2.5992e-09	3.4114e-08	2.0777e-08	6.6713e-12	1.9801e-08
3.6880e-09	1.0349e-07	8.9622e-11	2.4864e-12	3.2994e-08
4.1638e-09	1.3115e-07	1.1391e-08	9.2658e-13	2.4020e-08
3.5192e-09	8.8256e-08	4.5525e-08	3.4524e-13	2.3300e-08
1.9765e-09	8.5174e-08	8.4066e-08	1.2857e-13	1.8418e-08
1.0076e-09	6.0521e-08	9.8457e-08	4.7856e-14	5.9504e-09

Table 2: Comparison of absolute error for different values of h for problem1

h=1/12	h=1/20	h=1/40	h=1/80	h=1/40 (7 int points)
0.5193	0.2411	0.0377	0.1903	0.0392
0.1935	0.0899	0.0140	0.0709	0.1292
0.0721	0.0335	0.0052	0.0264	0.0482
0.0269	0.0125	0.0019	0.0099	0.0180
0.0100	0.0046	0.0007	0.0037	0.0067
0.0037	0.0017	0.0003	0.0014	0.0025
0.0014	0.0006	0.0001	0.0005	0.0009
0.0005	0.0002	0.0000	0.0002	0.0003
0.0002	0.0001	0.0000	0.0001	0.0001
0.0001	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000

Problem2: Consider one dimensional heat advection equation of the form

$$\frac{\partial C}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 C}{\partial x^2} \qquad for \;\; 0 < x < 1 \,, t > 0$$

with initial condition

$$C(x, 0) = \sin(\pi x)$$

and boundary conditions

$$C\,(0,t) = C(1,t) = 0.$$

The exact solution of above equation is given by $C(x,t) = \sin(\pi x)$.The above problem is solved and compared for different values of h which are shown in table 3. The error analysis in terms of absolute error for above problem for different h is given in table 4.

Table 3: Comparison of numerical values for different values of h for problem 2

h=1/12	**h=1/20**	**h=1/40**	**h=1/80**	**h=1/40 (7 int points)**
8.6604e-01	5.8779e-01	3.0902e-01	1.5643e-01	3.0748e-01
7.8361e-01	5.3199e-01	2.7969e-01	1.4159e-01	9.5230e-05
7.0902e-01	4.8136e-01	2.5306e-01	1.2810e-01	7.0877e-09
6.4154e-01	4.3550e-01	2.2896e-01	1.1590e-01	-7.3630e-10
5.8049e-01	3.9401e-01	2.0717e-01	1.0486e-01	3.8474e-10
5.2526e-01	3.5648e-01	1.8746e-01	9.4874e-02	2.2891e-09
4.7528e-01	3.2255e-01	1.6962e-01	8.5844e-02	1.4376e-09
4.3006e-01	2.9186e-01	1.5348e-01	7.7675e-02	2.5032e-10
3.8913e-01	2.6410e-01	1.3888e-01	7.0285e-02	9.7379e-11
3.5210e-01	2.3898e-01	1.2566e-01	6.3598e-02	-1.1101e-10
3.1859e-01	2.1624e-01	1.1370e-01	5.7548e-02	-1.2688e-10
2.8827e-01	1.9567e-01	1.0288e-01	5.2073e-02	-6.2996e-11
2.6083e-01	1.7705e-01	9.3087e-02	4.7117e-02	-1.1656e-11
2.3601e-01	1.6020e-01	8.4229e-02	4.2634e-02	5.0399e-13
2.1355e-01	1.4496e-01	7.6213e-02	3.8576e-02	6.4312e-12
1.9323e-01	1.3116e-01	6.8960e-02	3.4905e-02	1.3944e-11
1.7484e-01	1.1868e-01	6.2397e-02	3.1583e-02	1.6908e-11
1.5820e-01	1.0738e-01	5.6459e-02	2.8577e-02	1.5322e-11
1.4314e-01	9.7165e-02	5.1086e-02	2.5857e-02	9.1878e-12
1.2952e-01	8.7918e-02	4.6224e-02	2.3397e-02	1.0095e-12
1.1719e-01	7.9551e-02	4.1825e-02	2.1170e-02	5.8025e-13
1.0603e-01	7.1980e-02	3.7845e-02	1.9155e-02	2.1572e-13
9.5938e-02	6.5129e-02	3.4244e-02	1.7332e-02	-8.4090e-14
8.6807e-02	5.8930e-02	3.0986e-02	1.5682e-02	-3.1919e-13
7.8544e-02	5.3321e-02	2.8038e-02	1.4189e-02	-4.8151e-13
7.1069e-02	4.8245e-02	2.5370e-02	1.2839e-02	-5.3952e-13
6.4304e-02	4.3653e-02	2.2956e-02	1.1617e-02	-5.1795e-13
5.8184e-02	3.9497e-02	2.0772e-02	1.0511e-02	-4.1681e-13
5.2646e-02	3.5738e-02	1.8796e-02	9.5104e-03	-2.3610e-13
4.7635e-02	3.2336e-02	1.7007e-02	8.6053e-03	-2.7125e-14
4.3101e-02	2.9259e-02	1.5389e-02	7.7862e-03	9.5643e-15

Table 4: Comparison of absolute error for different values of h for problem 2

h=1/12	h=1/20	h=1/40	h=1/80	h=1/40 (7 int points)
0.05193	0.02411	0.00377	0.01903	0.03921
0.00239	0.01276	0.02315	0.02884	0.03467
0.02264	0.02651	0.03038	0.03250	0.03467
0.03019	0.03163	0.03307	0.03386	0.03467

0.03300	0.03354	0.03407	0.03437	0.03467
0.03405	0.03425	0.03445	0.03456	0.03467
0.03444	0.03451	0.03459	0.03463	0.03467
0.03458	0.03461	0.03464	0.03466	0.03467
0.03464	0.03465	0.03466	0.03466	0.03467
0.03466	0.03466	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
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0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467
0.03467	0.03467	0.03467	0.03467	0.03467

Problem 3: Consider one dimensional heat advection equation with diffusion term

$$\frac{\partial C}{\partial t} = (0.1)\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \qquad for\ 0 < x < 1\,, t > 0$$

with initial condition

$$C(x,0) = \exp(5x)\,[\cos\left(\frac{\pi}{2}x\right) + 0.25\sin\left(\frac{\pi}{2}x\right)]$$

and boundary conditions $C(0,t) = \exp(5(-\frac{t}{2}))\exp(-\frac{\pi^2}{40}t)$

and $C(1,t) = \exp(5(1-\frac{t}{2}))\exp(-\frac{\pi^2}{40}t)[0.25\sin\left(\frac{\pi}{2}\right)]$

The exact solution of above equation is given by $C(x,t) = \exp(5(x-\frac{t}{2}))\exp(-\frac{\pi^2}{40}t)[\cos\left(\frac{\pi}{2}x\right) + 0.25\sin\left(\frac{\pi}{2}x\right)]$. The above problem is solved by using method of orthogonal collocation on finite elements and compared at different grid points which are shown in table 5. The collocation points are taken as roots of Legendre Polynomial described as above.

Time(t)	Grid Points (h=0.08)								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	1.0452E-01	2.0794E-01	3.0906E-01	9.9452E-01	1.0000E+00	9.9452E-01	3.0906E-01	2.0794E-01	1.0452E-01
0.1	9.4611E-02	1.8819E-01	2.7969E-01	8.9989E-01	9.0484E-01	8.9989E-01	2.7969E-01	1.8819E-01	9.4611E-02
0.2	8.5605E-02	1.7028E-01	2.5307E-01	8.1423E-01	8.1871E-01	8.1423E-01	2.5307E-01	1.7028E-01	8.5605E-02
0.3	7.7458E-02	1.5407E-01	2.2898E-01	7.3674E-01	7.4080E-01	7.3674E-01	2.2898E-01	1.5407E-01	7.7458E-02
0.4	7.0088E-02	1.3941E-01	2.0719E-01	6.6664E-01	6.7031E-01	6.6664E-01	2.0719E-01	1.3941E-01	7.0088E-02
0.5	6.3419E-02	1.2615E-01	1.8748E-01	6.0321E-01	6.0653E-01	6.0321E-01	1.8748E-01	1.2615E-01	6.3419E-02
0.6	5.7385E-02	1.1415E-01	1.6964E-01	5.4581E-01	5.4882E-01	5.4581E-01	1.6964E-01	1.1415E-01	5.7385E-02
0.7	5.1924E-02	1.0329E-01	1.5350E-01	4.9388E-01	4.9660E-01	4.9388E-01	1.5350E-01	1.0329E-01	5.1924E-02
0.8	4.6983E-02	9.3456E-02	1.3889E-01	4.4688E-01	4.4934E-01	4.4688E-01	1.3889E-01	9.3456E-02	4.6983E-02
0.9	4.2512E-02	8.4562E-02	1.2567E-01	4.0435E-01	4.0658E-01	4.0435E-01	1.2567E-01	8.4562E-02	4.2512E-02
1.0	3.8466E-02	7.6514E-02	1.1371E-01	3.6587E-01	3.6788E-01	3.6587E-01	1.1371E-01	7.6514E-02	3.8466E-02
1.1	3.4805E-02	6.9232E-02	1.0289E-01	3.3105E-01	3.3287E-01	3.3105E-01	1.0289E-01	6.9232E-02	3.4805E-02
1.2	3.1493E-02	6.2643E-02	9.3098E-02	2.9954E-01	3.0119E-01	2.9954E-01	9.3098E-02	6.2643E-02	3.1493E-02
1.3	2.8495E-02	5.6681E-02	8.4238E-02	2.7103E-01	2.7252E-01	2.7103E-01	8.4238E-02	5.6681E-02	2.8495E-02
1.4	2.5784E-02	5.1287E-02	7.6221E-02	2.4524E-01	2.4659E-01	2.4524E-01	7.6221E-02	5.1287E-02	2.5784E-02
1.5	2.3330E-02	4.6406E-02	6.8967E-02	2.2190E-01	2.2312E-01	2.2190E-01	6.8967E-02	4.6406E-02	2.3330E-02
1.6	2.1109E-02	4.1990E-02	6.2404E-02	2.0078E-01	2.0189E-01	2.0078E-01	6.2404E-02	4.1990E-02	2.1109E-02
1.7	1.9100E-02	3.7994E-02	5.6465E-02	1.8167E-01	1.8267E-01	1.8167E-01	5.6465E-02	3.7994E-02	1.9100E-02
1.8	1.7282E-02	3.4377E-02	5.1090E-02	1.6438E-01	1.6529E-01	1.6438E-01	5.1090E-02	3.4377E-02	1.7282E-02
1.9	1.5637E-02	3.1105E-02	4.6227E-02	1.4873E-01	1.4955E-01	1.4873E-01	4.6227E-02	3.1105E-02	1.5637E-02
2.0	1.4149E-02	2.8144E-02	4.1826E-02	1.3458E-01	1.3532E-01	1.3458E-01	4.1826E-02	2.8144E-02	1.4149E-02
2.1	1.2802E-02	2.5465E-02	3.7845E-02	1.2177E-01	1.2244E-01	1.2177E-01	3.7845E-02	2.5465E-02	1.2802E-02
2.2	1.1583E-02	2.3041E-02	3.4243E-02	1.1017E-01	1.1078E-01	1.1017E-01	3.4243E-02	2.3041E-02	1.1583E-02
2.3	1.0481E-02	2.0848E-02	3.0984E-02	9.9689E-02	1.0024E-01	9.9689E-02	3.0984E-02	2.0848E-02	1.0481E-02
2.4	9.4833E-03	1.8864E-02	2.8034E-02	9.0199E-02	9.0696E-02	9.0199E-02	2.8034E-02	1.8864E-02	9.4833E-03
2.5	8.5807E-03	1.7068E-02	2.5366E-02	8.1615E-02	8.2065E-02	8.1615E-02	2.5366E-02	1.7068E-02	8.5807E-03
2.6	7.7640E-03	1.5444E-02	2.2952E-02	7.3846E-02	7.4253E-02	7.3846E-02	2.2952E-02	1.5444E-02	7.7640E-03
2.7	7.0250E-03	1.3974E-02	2.0767E-02	6.6818E-02	6.7186E-02	6.6818E-02	2.0767E-02	1.3974E-02	7.0250E-03
2.8	6.3563E-03	1.2644E-02	1.8791E-02	6.0458E-02	6.0791E-02	6.0458E-02	1.8791E-02	1.2644E-02	6.3563E-03
2.9	5.7514E-03	1.1440E-02	1.7002E-02	5.4704E-02	5.5005E-02	5.4704E-02	1.7002E-02	1.1440E-02	5.7514E-03
3.0	5.2039E-03	1.0351E-02	1.5384E-02	4.9496E-02	4.9769E-02	4.9496E-02	1.5384E-02	1.0351E-02	5.2039E-03

PROBLEM 1

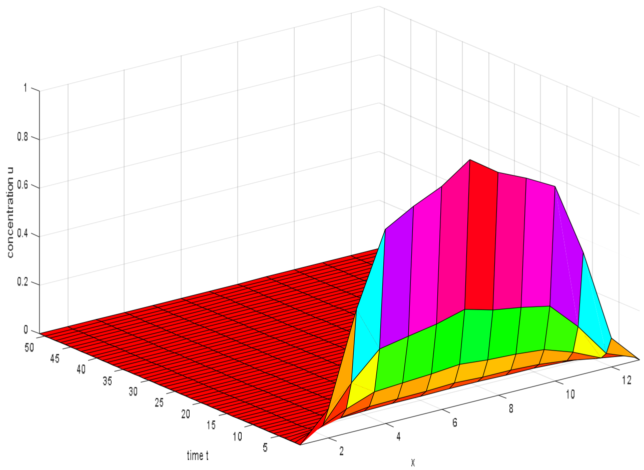

Fig.(1). OCFE solution for h=1/12

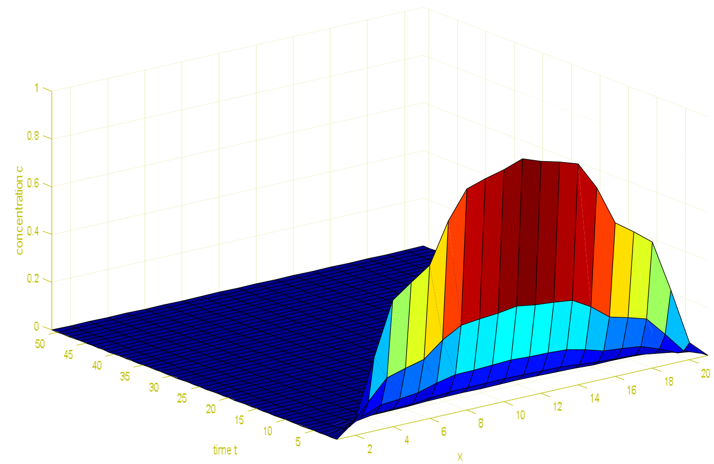

Fig.(2). OCFE solution for h=1/20

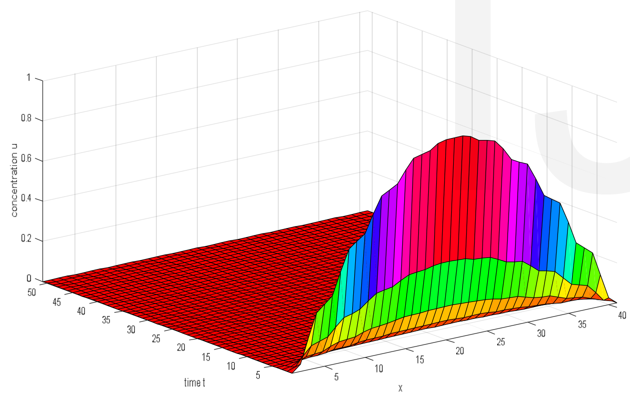

Fig.(3). OCFE solution for h=1/40

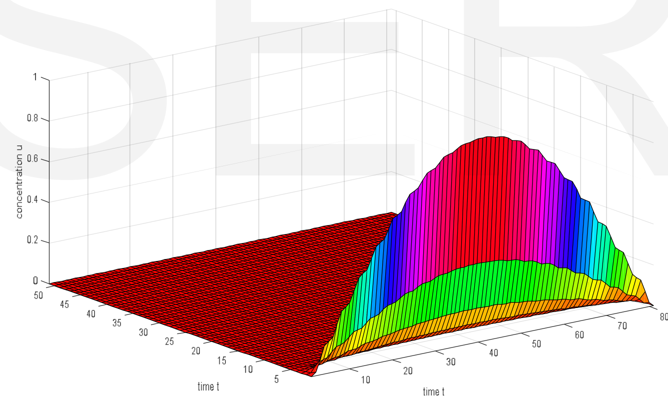

Fig.(4). OCFE solution for h=1/80

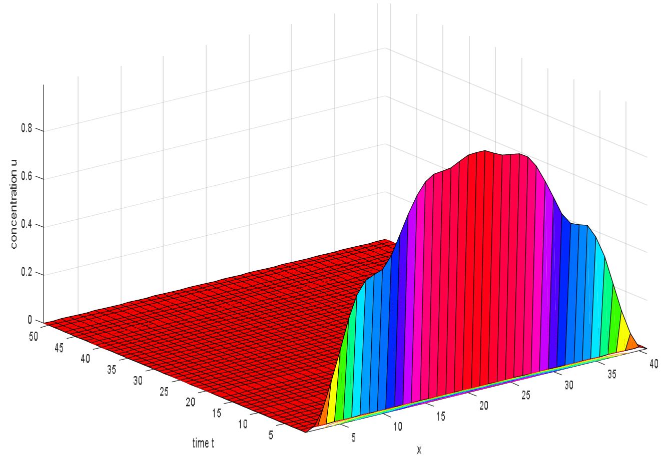

Fig.(5). OCFE solution for h=1/40(7 int points)

PROBLEM 2:

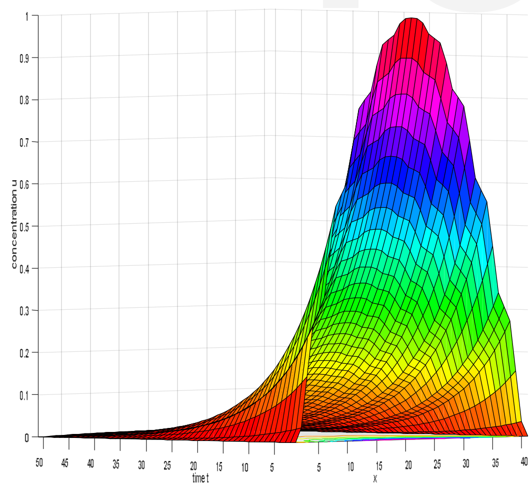

Fig.(6). OCFE solution for h=1/12

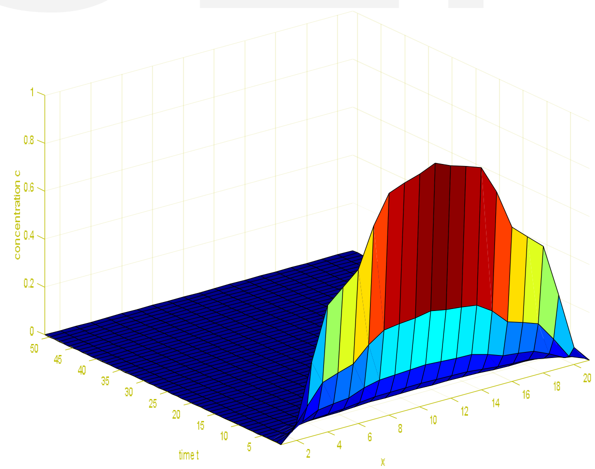

Fig.(7). OCFE solution for h=1/20

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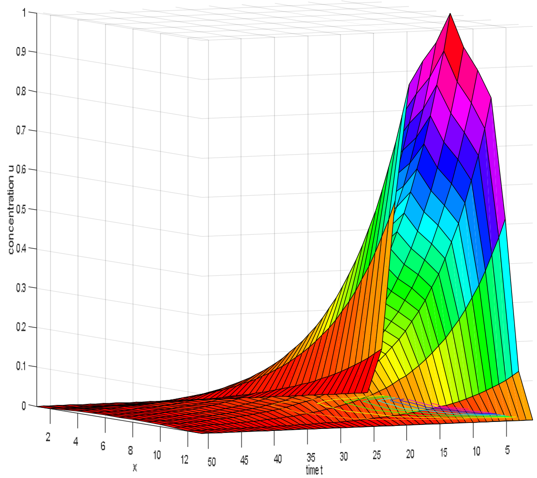

Fig.(8). OCFE solution for h=1/40

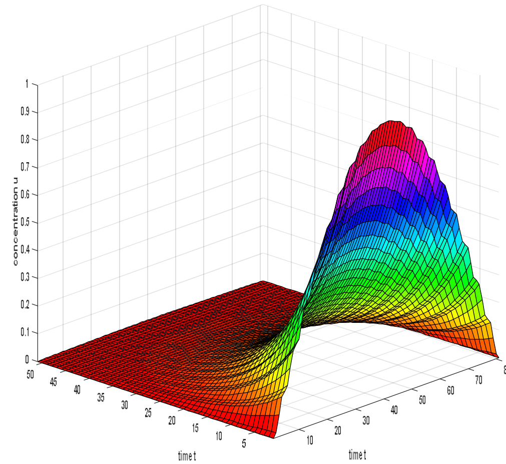

Fig.(9). OCFE solution for h=1/80

Conclusion:

Heat and advection-diffusion equations are solved numerically by using Orthogonal collocation on finite elements. Numerical values obtained are in good agreement with exact ones. It can be seen that the method of orthogonal collocation on finite elements gives results in less computational time with good efficiency. The results are verified for three test problems. Thus the numerical technique of orthogonal collocation on finite elements is simple and efficient for the solution of heat and advection equations.

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